

The Black Hole Quantum Entropy and Its Minimal Value

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Abstract

In the paper it is demonstrated that the Schwarzschild black-hole quantum entropy computed within the scope of the Generalized Uncertainty Principle (GUP) has a nonzero minimum under the assumption that for a radius of the black hole the lower limit is placed, whose value is twice the minimal length. Such a limit is quite natural when using, as a proper deformation parameter in a quantum theory with a minimal length, the dimensionless small parameter introduced previously by one of the authors in co-authorship with his colleagues and caused by modification of the density matrix at Planck's scales. The results obtained have been compared to the results of other authors and analyzed from the viewpoint of their compatibility with the well-known facts and the holographic principle in particular.

1 Introduction

The black hole entropy is essential for a quantum field theory in the curved space-time, for cosmology, physics of the early Universe, and the like since the appearance of the renowned works by Bekenstein and Hawking in the seventies of the last century [1],[2]. Bekenstein and Hawking have introduced the black hole entropy in a semiclassical approximation only, i.e. when material fields are quantized on the basis of the classical space-time. Then the following familiar formula for the black hole entropy is the case:

$$S^{BH} = \frac{A}{4l_p^2}. \quad (1)$$

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But it is obvious that a change to higher energies (with regard to the quantum-gravitational effects) necessitates addition of quantum corrections in the right-hand side of (1). The importance of this problem has been greatly increased with the introduction of the Generalized Uncertainty Principle (GUP)

$$\Delta x \geq \frac{\hbar}{\Delta p} + \ell^2 \frac{\Delta p}{\hbar}, \quad (2)$$

where $\ell^2 = \lambda l_p^2$ and λ – dimensionless numerical factor.

GUP, first brought about in a superstring theory [3], has been supported by the findings in the fields of fundamental research having no relation to the superstring theory [4]–[10].

Apparently, GUP (2) leads to a minimal length on the order of Planck's length

$$\Delta x_{min} = 2\sqrt{\lambda} l_p \quad (3)$$

and, moreover, is responsible for the appearance of the quantum corrections in the right-hand side of (1).

At the present time these corrections are quite clearly understood (for example, see [11]–[13]), however, leaving some unanswered questions, the most interesting of which are listed below.

(1) As quantum theory with GUP is a deformed quantum theory [14] with the corresponding deformation of the Heisenberg algebra [9],[10], there is a problem concerning the dependence of the final result on selection and variability domain of the deformation parameter.

(2) The same problem concerns the entropy of micro(black)holes or planckian black holes when we proceed to the region of planckian energies.

In this work it is shown that with the use of α , introduced in [15]–[27], as a deformation parameter a change from lower to higher energies and vice versa is quite natural, meeting the Hooft-Susskind Holographic Principle [28]–[30], on the one hand, and the notion of a nonzero minimal energy is also reasonable, on the other hand.

2 GUP and Quantum Corrections to Semi-classical Formula of BH Entropy

As noted in the previous Section, a quantum theory extended by inclusion of GUP is deformed and associated with a deformation parameter. Of course, selection of this parameter is inconclusive (for example, [9]).

In [25]–[27] it has been demonstrated that, at least, for the simplest variant of GUP – merely deformation of the commutator $[\vec{x}, \vec{p}]$:

$$[\vec{x}, \vec{p}] = i\hbar(1 + \beta^2 \vec{p}^2 + \dots) \quad (4)$$

the parameter $\alpha_x = l_{min}^2/x^2$, where x is the measuring scale, is also the GUP-deformation parameter, and (4) may take the following form (for example, [26], p. 943):

$$[\vec{x}, \vec{p}] = i\hbar(1 + a_1\alpha_x + a_2\alpha_x^2 + \dots). \quad (5)$$

Note that α_x may be introduced into a quantum field theory with the minimal length l_{min} , regardless of the origin of this minimal length (GUP or not) [15]–[21].

At the same time, there is one important feature of using the parameter α_x in a theory with the fundamental length (ultraviolet cutoff). The variability domain of this parameter is $0 < \alpha_x \leq 1/4$ [16]–[21] and hence, with this approach based on the high-energy density-matrix deformation, apart from the **minimal length** l_{min} , we have the **minimal measurable length** l_{min}^{meas}

$$l_{min}^{meas} = 2l_{min}. \quad (6)$$

So, α_x is small (for the known energies very small) and dimensionless. Only at Planck's scales we have

$$\lim_{x \rightarrow l_{min}^{meas}} \alpha_x = \frac{1}{4}. \quad (7)$$

The parameter α_x is very convenient in studies of the black hole thermodynamic characteristics and specifically of the entropy S .

We use the results presented in the work [11] that is now canonical. The work gives the most complete formula for the Schwarzschild black-hole quantum entropy within the scope of GUP that takes the following form ([11], formula (16)):

$$S_{GUP}^{BH} = \frac{A}{4l_p^2} - \frac{\pi\lambda}{4} \ln \left(\frac{A}{4l_p^2} \right) + \sum_{n=1}^{\infty} c_n \left(\frac{A}{4l_p^2} \right)^{-n} + \text{const}, \quad (8)$$

where the expansion coefficients $c_n \propto \lambda^{n+1}$ can always be computed to any desired order of accuracy.

Then, for a black hole of the radius R , in terms of α_R (8) is as follows:

$$S_{GUP}^{BH}[\alpha_R] = 4\pi\lambda\alpha_R^{-1} + \frac{\pi\lambda}{4}\ln\alpha_R + \sum_{n=1}^{\infty} \frac{c_n}{(4\pi\lambda)^n} \alpha_R^n + \text{const}' . \quad (9)$$

Here the integration constant is redefined as $\text{const} - > \text{const}' + \frac{\pi\lambda}{4}\log\lambda$.

Let us find an extremum of $S_{GUP}^{BH}[\alpha_R]$ in the interval $0 < \alpha_R \leq 1/4$ for different values of the numerical factor λ . It is clear that this function has no maximum as for $\alpha_R \rightarrow 0$ (the case of large black holes) the first (semi-classical) term $\propto \alpha_R^{-1}$ becomes infinitely large to suppress the remaining "tail" in the right-hand side (9). Because of this, of particular importance is finding of a minimum $S_{GUP}^{BH}[\alpha_R]$.

As seen, the minimal element of area

$$(\Delta A)_{min} \simeq \epsilon l_p^2 \Delta p \Delta x \quad (10)$$

$$\Delta p \simeq \frac{\Delta x}{2\lambda l_p^2} \left[1 - \sqrt{1 - \frac{4\lambda l_p^2}{\Delta x^2}} \right] \quad (11)$$

is a strictly positive value. Assuming that the entropy of such an area is equal to $b = \ln 2$, for the total entropy we can write the following equation:

$$\frac{dS}{dA} \simeq \frac{(\Delta S)_{min}}{(\Delta A)_{min}}. \quad (12)$$

On going from the area to the parameter $\alpha_R = \frac{16\pi\lambda l_p^2}{A}$ this equation takes a simple form

$$\frac{dS}{d\alpha_R} = -\pi\lambda \frac{2 + \sqrt{4 - \alpha_R}}{\alpha_R^2} \quad (13)$$

The right-hand side of the equation is obviously negative for all $\alpha_R < 4$ and for $\lambda > 0$. Consequently, the entropy is a monotonically decreasing function of the parameter α_R .

From the condition $\alpha_R \in (0, 1/4]$ it follows that the entropy is at its minimum at the point $\alpha_R = 1/4$.

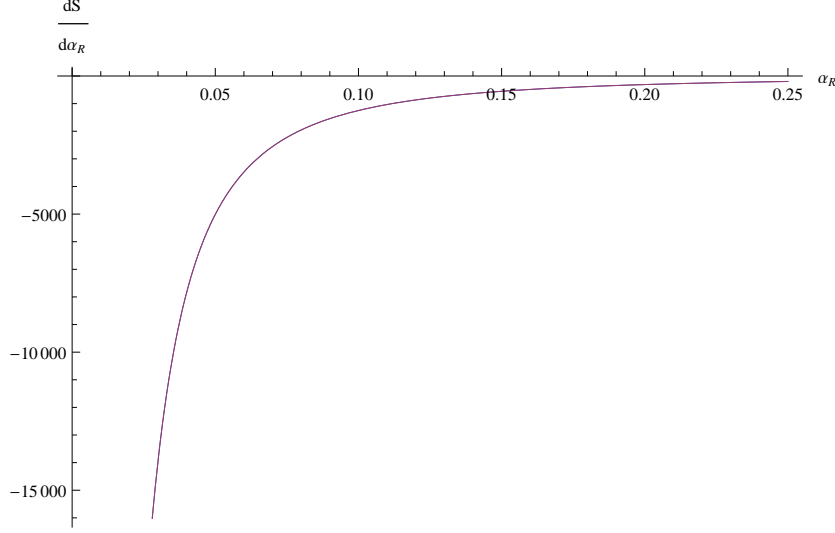


Figure 1: Derivative of the entropy $\frac{dS}{d\alpha_R}$

Integrating the equation, we get

$$S = \pi\lambda \left(\frac{4}{\alpha_R} + \frac{\log(\alpha_R)}{4} + \frac{\alpha_R}{64} + \frac{\alpha_R^2}{1024} + \frac{5\alpha_R^3}{49152} + \dots + C \right). \quad (14)$$

The integration constant C is fixed in accordance with [33] from the condition of zero entropy at a minimal area of the event horizon $A_p = 4\pi(\Delta x_{min})^2 = 16\pi\lambda l_p^2$. This complies with the parameter value $\alpha_R = 1$. We have derived $C = -4.01672$. But a minimal *measurable* area is associated with $\alpha_R = 1/4$. This leads to the existence of a minimal entropy that equals $S_{min} = 36.5703\lambda$.

Performing similar computations for another type of GUP

$$\Delta x \Delta p \geq \sqrt{1 + 2\lambda l_p^2 \Delta p^2}, \quad (15)$$

we obtain

$$\frac{dS}{d\alpha_R} = -4\pi\lambda \frac{\sqrt{1 - \frac{\alpha_R}{8}}}{\alpha_R^2}, \quad (16)$$

$$S = \pi\lambda \left(\frac{4}{\alpha_R} + \frac{\log(\alpha_R)}{4} + \frac{\alpha_R}{128} + \frac{\alpha_R^2}{4096} + \frac{5\alpha_R^3}{393216} + \dots + C \right). \quad (17)$$

Here the integration constant is $C = -4.008007$, and a minimal entropy (for $\alpha_R = 1/4$) is equal to $S_{min} = 36.5911\lambda$.

It is seen that the difference between these two cases is insignificant making it possible to infer: the uncertainty associated with the coefficients c_n is negligible.

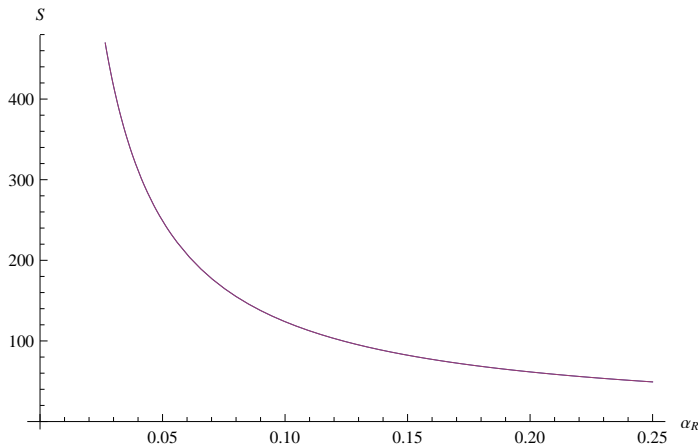


Figure 2: The entropy S as a function of α_R

3 Analysis and Comments

By our approach, for Schwarzschild black holes there is a **minimal nonzero entropy**, $S_{min}^{bh} > 0$ coming as an attribute for a micro-black hole of the radius

$$r_{bh}^{min} \approx 2l_{min}. \quad (18)$$

This result is consistent with the results of other authors, in particular, in the works [31], [32], where it has been demonstrated that the final stage of a black hole is the planckian remnant with zero entropy and zero temperature. However, this occurs in the case of the noncommutative space-time, i.e. when the following condition is fulfilled for operators of the space coordinates:

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}, \quad (19)$$

where $\theta^{\mu\nu}$ is an anti-symmetric matrix.

This restriction is very strong but it is not imposed in the present paper. As shown in ([33],section 2), the simplest variant of GUP (without the

noncommutativity condition (19)) may finally result in a zero entropy for the planckian black-hole remnant ([33], formulae (21),(22)). However, in [33], as distinct from (18), the consideration is given to the limiting case when

$$r_{bh}^{min} = l_{min} = l_p. \quad (20)$$

The cutoff introduced due to (6), (7) factors out this case and hence there is no conflict with the result of [33].

The viewpoint of this work, where the considered “minimal” black holes have a radius that is equal to the minimal measurable length $l_{min}^{meas} = 2l_{min}$ (6)) rather than to the minimal length l_{min} , offers particular advantages summarized below.

I. This viewpoint enables one to hold for such holes the holographic principle [28]–[30](though within the quantum corrections). It is clear that the zero-entropy planckian remnants violate this principle cardinally.

II. Besides, from the viewpoint of a quantum information theory, such remnants are as good as empty space, though having great curvature, and this is very odd.

III. It is known that micro-black holes are important elements of the presently available models for the space-time foam (for example, [34]) but, proceeding from item II, these models are liable to be problematic.

IV. Finally, at least by the heuristic approach, it has been demonstrated that the Holographic Principle is following from the space-time foam models (for example, [35]), i.e. from Planck’s energies. Because of this, taking into consideration the inferences of items I–III, it is desirable to hold this Principle in some or other form at Planck’s scales as well.

4 Conclusion

If evaporation of a black hole results finally in the planckian remnant with zero energy and temperature, in addition, this means that Einstein’s equation for the spherically-symmetric horizon space, close to horizon written as a thermodynamic identity (first law of thermodynamics) ([36] formula

(119)), in principle has no ultraviolet limit (cutoff) as in this limit the left-hand side of equation (119) in [36] becomes degenerate. As shown in [27], when such a limit is considered in terms of the deformation parameter α_R and its corresponding variability domain $0 < \alpha_R \leq 1/4$ (then called the **measurable ultraviolet limit**), it is applicable within the scope of both equilibrium and nonequilibrium thermodynamics. .

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